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OPTIMAL CAPITAL POLICY WITH IRREVERSIBLE INVESTMENT

BY

KENNETH J. ARROW

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INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
SERRA HOUSE, STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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0. Introduction

It is Sir John Hicks's Value and Capital which taught us clearly the formulation of capital theory as an optimization problem for the firm. He set a general framework within which all subsequent work has taken place.

If one may generalize a bit crudely, the principal subsequent innovation has been the more explicit recognition of the recursive nature of the production process. As a rough empirical generalization, the links between inputs and outputs at different points of time are built up out of links between successive time points. In discrete-time (period) analysis, this means that outputs at time $t+1$ are determined directly by inputs at time t, independent of earlier inputs; the latter may still have an indirect influence by affecting the availability of the inputs at time t. In continuous-time analysis, which will be employed here, the basic production relation is between the stocks of capital goods and the flows of current inputs and outputs; the earlier past is controlling only in that the stock of capital goods is a cumulation of past flows.

This recursive aspect of the production process simplifies analysis and computation, as was first recognized in the context of inventory theory in the magisterial work of Massé (1946) (unfortunately ignored in the English-language literature) and independently by Arrow, Harris, and Marschak (1951). Subsequently, the mathematician Bellman (1957)

recognized the basic principle of recursive optimization common to inventory theory, sequential analysis of statistical data, and a host of other control processes in the technological and economic realms and developed the set of computational methods and principles known as dynamic programming. Finally, the Russian mathematician Pontryagin and his associates (1962) developed an elegant theory of control of recursive processes related both to Bellman's work and to the classical calculus of variations. The Pontryagin principle, which will be used in this paper, has the great advantage of yielding economically interesting results very naturally.

This paper follows several others investigating under various hypotheses the optimal policy of a firm with regard to the holding of fixed capital (Arrow, Beckmann, and Karlin, 1958; Arrow, 1962; Nerlove and Arrow, 1962; Arrow, 1964). Assume, for simplicity, that there is only one type of capital good, all other inputs and outputs being flows. Then for any fixed stock of capital goods there is at any moment a most profitable current policy with regard to flow variables; we assume the flow optimization to have taken place and therefore have defined a function relating operating profits (excess of sales over costs of flow inputs) to the stock of capital goods. This function may, however, shift over time because of shifts in technological relations and demand and supply conditions (in the case of a monopoly there may be shifts in the demand curve; for a monopsony, shifts in the supply curves of the factors; for a competitive firm, demand or supply shifts are simply changes in output or input prices). The cash flow at any moment is the difference between operating profits and gross investment.

We assume throughout a perfect capital market so that the aim of the firm is to maximize the integral of discounted cash flows, where the discounting is done at the market rates of interest (which may be changing over time).

The problem assumes different forms according to the assumption made about the cost schedule of capital goods to the firm. To assume that there is a truly perfect capital goods market to the firm means that, at any moment of time, there is a fixed price of capital goods at which the firm can buy or sell in any magnitude. In that case, the optimal policy has a special "myopic" property (see section 1), which is obvious enough once observed, but which has only recently been given much weight in the literature.

From a realistic point of view, there will be many situations in which the sale of capital goods cannot be accomplished at the same price as their purchase. There are installation costs, which are added to the purchase price but cannot be recovered on sale; indeed, there may on the contrary be additional costs of detaching and moving machinery. Again, sufficiently specialized machinery and plant may have little value to others. So resale prices may be substantially below replacement costs. For simplicity we will make the extreme assumption that resale of capital goods is impossible, so that gross investment is constrained to be non-negative. It is clear that this may affect investment policy strongly. Obviously, at a point where a firm would like to sell capital goods at the going price if it could, it will be barred from this disinvestment. More subtly, at a time which investment is still profitable as far as current calculations are concerned, the firm may refrain from investment if it anticipates that in the relatively near future it would have

disinvested if it could. It is this problem which will be studied in the present paper.[1]

In section 1 we briefly remind the reader of the myopic optimization rule for the case where investment is costlessly reversible. In section 2 we state the model more fully, and note that the case of exponential depreciation can be reduced to that of zero depreciation, which will be considered henceforth. Section 3 characterizes the solution in the case of diminishing returns, and section 4 indicates how the optimum might be effectively computed. In section 5 the case of constant returns to capital is analyzed. Finally, section 6 is devoted to some remarks on possible empirical implications.

1. The Case of Reversible Investment

A rigorous analysis of this case has been carried out in the earlier publications cited and will follow as a special case of the general reversible case. But the result is obvious once stated: at each instant, hold that stock of capital for which the marginal profitability of capital equals the cost of capital, by which is meant the sum of the short-term interest rate, depreciation, and the rate of decline in capital goods prices.

Once stated, the rule will seem banal, for it is one of the standard equations of capital theory. Yet its implications are often ignored. It means that the decision as to the stock of capital to be held at any instant of time is myopic, being independent of future developments in

[1] Arrow, Beckmann, and Karlin (1958) studied in detail a very special case of the present results in which all prices are fixed and there is a fixed-capital-output ratio so that revenue is proportional to the smaller of the two quantities, capacity and demand, the latter varying over time. Nerlove and Arrow (1962) dealt with advertising, considered in effect as a capital good. Advertising is clearly irreversible, but it was then not possible to treat the optimal policy in a fully adequate fashion.

technology, demand or anything else; forecasts for only the most immediate future are needed and then only as to capital goods prices.[2] The argument for this rule is simple: When investment is reversible, then the firm can buy a unit of capital goods, use it and derive its marginal product for an arbitrarily short time span, and then sell the undepreciated portion, possibly at a different price.

This rule defines a demand function for capital very different from Keynes's marginal efficiency of capital (1936, Chapter 11), unless the latter is so interpreted as to make it meaningless. In effect, Keynes's rule is that an investment is justified if and only if the sum of discounted returns at least equals the cost. But the return to a particular investment is not a datum but depends on the total volume of capital in the system. With reversible investment, a natural interpretation is that the given investment is taken to be the marginal investment in every future period, in which case the return in any future period is simply the marginal productivity of capital, net of depreciation and price changes, at that time. <u>If</u> the firm follows in the future the myopic rule of equating marginal productivity of capital to the rate of interest, then it is true that the Keynes rule amounts to accepting the same rule today. But the interest of the marginal efficiency concept evaporates, since all the content lies in the myopic rule.

The marginal efficiency rule can easily be misunderstood to mean that in evaluating future returns the new investment today is to be regarded as marginal not to a future optimal stock but to the present stock. This is wrong, however. Suppose, for example, the marginal productivity schedule is anticipated to be shifting upward over time.

[2] Strictly speaking, if depreciation is not exponential, the depreciation term does depend to some extent on the future course of interest rates; for the details, see Arrow (1964).

Then the sum of discounted profits may exceed the cost because of high returns in the distant future, even though the marginal product may be less than interest in the near future. It is indeed true that it is better to undertake the investment than not to, but these are not the only alternatives. A still better one would be to postpone the investment until it is profitable in the immediate future. It is this possibility of postponement which justifies the myopic rule.[3]

2. Explicit Formulation of the Model

The model will be formulated very generally in some respects at least. We assume one type of capital good, the stock of which is denoted by K. At each moment of time the operating profit function, $P(K,t)$, denotes the profits obtainable from a given stock of capital, K, by optimal employment of other factors. Thus, the variations of P over time may reflect changes in technology, supply conditions for other factors, or demand conditions. The firm faces a perfect market for liquid capital, but the interest rates may be changing in time in a known way. Let $\alpha(t)$ be the rate at which receipts at time t are discounted back to time 0, the beginning of the optimization period. Then the (short-term) interest rate at time t is

$$\rho(t) = -\dot{\alpha}(t)/\alpha(t),$$

where the dot denotes differentiation with respect to time.

We will take the price of capital goods as numeraire. This implies

[3] For other recognitions of the myopic rule for capital policy, see Marglin (1963, pp. 20-27), Kurahashi (1963), and Champernowne (1964, p. 185). Fisher's "rate of return over cost" (1930, pp. 155-58) appears to be similar to Keynes's "marginal efficiency of capital" but in fact Fisher considered returns and costs to be measured relative to the best alternative. If the alternative of postponement is considered, then the myopic rule is derivable, though Fisher does not do so explicitly.

that the interest and discount rates are expressed in terms of capital goods, rather than money, so that the interest rate used here is the money rate of interest less the rate of appreciation of capital goods prices.

Let $I(t)$ be the rate of gross investment. Then the cash flow at time t, in terms of capital goods, is $P(K_t,t) - I(t)$,[4] and therefore the sum of discounted returns is

$$(1) \qquad \int_0^{\infty} \alpha(t)[P(K_t,t) - I(t)]dt.$$

The aim of the firm is to maximize (1) by suitable choice of investment policy, $I(t)$. The evolution of the capital stock, $K(t)$, is determined by its initial value, $K(0)$, and by the investment policy. If we assume depreciation at a fixed exponential rate, δ, then

$$(2) \qquad \dot{K} = I - \delta K.$$

Finally, the assumption of irreversible investment means that gross investment must be non-negative.

$$(3) \qquad I(t) \geqq 0.$$

We also postulate positive but strictly diminishing returns to capital at any moment of time.

$$(4) \qquad P_K > 0, \; P_{KK} < 0.$$

In the remainder of this section it will be argued that this problem can be transformed into another of the same form, but with $\delta = 0$. Then for the remainder of the paper we can assume the absence of depreciation with no loss of generality.

[4] Functions of time, for example, $K(t)$, will also be symbolized by using t as a subscript, e.g., K_t, when typographically convenient, and the time variable may even be suppressed when its presence is clear from the context.

Let

$$x(t) = K(t)e^{\delta t},\ y(t) = I(t)e^{\delta t},\ \beta(t) = \alpha(t)e^{-\delta t},$$

$$\overline{P}(x,t) = e^{\delta t}P(xe^{-\delta t},t).$$

Then it is easy to calculate that

$$\dot{x} = (\dot{K} + \delta K)e^{\delta t} = I(t)e^{\delta t} = y,$$

and

$$\beta(t)[\overline{P}(x,t) - y(t)] = \alpha(t)[P(K,t) - I(t)],$$

so that the original problem is transformed into the maximization of

$$\int_0^\infty \beta(t)[\overline{P}(x,t) - y]dt,$$

subject to the accumulation condition, $\dot{x} = y$, the initial condition, $x(0) = K(0)$, and the non-negativity condition, $y \geqq 0$, following from (3). The new problem is indeed of the original form, with x and y replacing K and I, respectively, and $\delta = 0$. Further,

$$\overline{P}_x = P_K,\ \overline{P}_{xx} = e^{-\delta t}P_{KK},$$

so that conditions (3) still remain valid.

3. The Characterization of the Solution

We now assume that $\delta = 0$; then we wish to maximize

(1) $$\int_0^\infty \alpha(t)[P(K,t) - I(t)]dt,$$

subject to

(2) $$\dot{K} = I,$$

K(0) given, and

(3) $$I \geqq 0.$$

The return to an investment at any moment of time has two parts: the current cash flow and an addition to the sum of discounted future benefits. The latter is equal to the value of a gift of a unit of capital at time t. Let $p(t)$ be this shadow price of capital at time t, discounted back to time 0. Then the (discounted) value of investment at a given time t is

$$(4) \qquad H = \alpha(t)[P(K,t) - I] + p(t)I.$$

H is known in control theory as the Hamiltonian (after the 19th Century mathematician who introduced the concept). Then I is chosen so as to maximize H, subject to the condition (3).

However, we have to have a principle for determination of the shadow price of capital, p. Pontryagin and associates have shown that $p(t)$ must evolve in time, according to the differential equation

$$(5) \qquad \dot{p} = - \partial H/\partial K.$$

Let us rewrite (4) by setting

$$(6) \qquad q(t) = p(t) - \alpha(t);$$

then

$$(7) \qquad H = \alpha(t)\, P(K,t) + q(t)I.$$

The maximization of H with respect to I has a rather trivial form. If $q(t) < 0$, then the optimum I, subject to (3), must be 0. If $q(t) = 0$, then the optimum I can be any non-negative quantity. This does not mean that I is indeterminate; as will be seen shortly, it is determined by other considerations but not by the requirement that the Hamiltonian be maximized. If $q(t)$ were positive, then there would

be no optimum for I; the larger the better. This is, however, incompatible with the existence of an optimal policy, for $q(t)$ would be positive over an interval, and infinite investment over an interval is obviously non-optimal. The point is that if a policy of capital accumulation led to such a situation, it would have been better to have invested more earlier, so that the policy followed was non-optimal. We conclude:

(8) $\quad q(t) \leqq 0$; if $q(t) < 0$, then $I(t) = 0$.

The economic interpretation of (8) is straightforward. The comparison of q with 0 is, according to (6), a comparison of p with α; since the market price of capital goods is always 1, by the choice of numeraire, $\alpha(t)$ is the market price of capital goods discounted back to time 0, while $p(t)$ is the shadow price (the value of future benefits) similarly discounted. If q is negative, the shadow price is less than the market price, and it does not pay to invest. If $q = 0$, one is just indifferent at the margin between investing and not investing. If q could be positive, it would pay to invest infinitely, but it would also have paid to invest infinitely at an earlier time.

From (7), (5) becomes

(9) $\quad \dot{p} = -\alpha(t) P_K$,

which can also be written

$$\alpha(t) P_K + \dot{p} = 0,$$

i.e., discounted current returns plus changes in discounted shadow value should be zero, a restatement of the familiar equilibrium relation for the holding of assets, but with shadow prices substituted for market prices.

Since the short-run optimum condition (8) has been written in terms of q rather than p, it is convenient to reformulate the differential equation (9). From the definition (6), and (9),

$$\dot{q} = \dot{p} - \dot{\alpha} = \alpha(t)[-P_K - (\dot{\alpha}/\alpha)] ,$$

or from the definition of $\rho(t)$,

$$\dot{q} = \alpha(t)[\rho(t) - P_K(K_t,t)] . \tag{10}$$

The integral of (10) is straightforward:

$$q(t_1) - q(t_0) = \int_{t_0}^{t_1} \alpha(t)[\rho(t) - P_K(K_t,t)]dt. \tag{11}$$

Formally, the solution has been completely described. We seek three functions of time, $K(t)$, $q(t)$, and $I(t)$, jointly satisfying the conditions (2), (3), (8), and (10), with $K(0)$ given. The initial value $q(0)$ has not been explicitly defined; it has to be such that all these conditions can jointly be satisfied. Primarily, it has to be sufficiently small so that the condition $q(t) \leqq 1$ holds.

However, a good deal more can be said about the structure of the solution. First, it is necessary to discuss the possibility of discontinuous jumps in the stock of capital. A jump in the stock of capital would require an infinite rate of investment, but from the point of view of the firm there is nothing difficult to comprehend; it is simply the acquisition of a block of capital goods at some instant of time. We will show that it is never optimal to have a jump in the stock of capital except possibly at the very initial point of time.

Since $K(t)$ is monotone increasing, both the left- and right-hand limits exist at every point. Let $K(t)$ have a discontinuity at $t = t_0$, where $t_0 > 0$, and let $K(t_0-0)$ and $K(t_0+ 0)$ be the left-hand and

right-hand limits, respectively. Since $K(t)$ is increasing, we must have

$$K(t_0 - 0) < K(t_0 + 0).$$

Define

(12) $\quad r(K,t) = \rho(t) - P_K(K,t).$

Since P_K is strictly decreasing in K,

(13) $\quad r(K,t)$ is strictly increasing in K for fixed t.

Then we can choose c_1 and c_2 so that

$$r[K(t_0 - 0),t_0] < c_1 < c_2 < r[K(t_0 + 0),t_0] .$$

Again, since $K(t)$ is increasing, we must have

$$K(t) \leqq K(t_0 - 0) \quad \text{for} \quad t < t_0, \; K(t) \geqq K(t_0 + 0) \quad \text{for} \quad t > t_0 ,$$

and therefore

$$r(K_t,t) \leqq r[K(t_0 - 0),t] \quad \text{for} \quad t < t_0 ,$$

$$r(K_t,t) \geqq r[K(t_0 + 0),t] \quad \text{for} \quad t > t_0 .$$

Finally, since $r(K,t)$ is continuous in t for fixed K,

$$r[K(t_0 - 0),t] < c_1 \quad \text{for } t \text{ sufficiently close to } t_0,$$

$$r[K(t_0 + 0),t] > c_2 \quad \text{for } t \text{ sufficiently close to } t_0.$$

Combine these statements and recall (10).

$$\dot{q}(t)/\alpha(t) < c_1 \quad \text{for} \quad t_0 - \epsilon < t < t_0,$$

$$\dot{q}(t)/\alpha(t) > c_2 \quad \text{for} \quad t_0 < t < t_0 + \epsilon ,$$

for some $\epsilon > 0$.

But since investment is taking place at time t_0, $q(t_0) = 0$; since

$q(t) \leqq 0$ everywhere, by (8), it is impossible that $\dot{q}(t) < 0$ for all t, $t_0 - \epsilon < t < t_0$, for then we would have $q(t) > q(t_0) = 0$ throughout that interval. Since $\dot{q}(t) \geqq 0$ for some t, $t_0 - \epsilon < t < t_0$, we must have $c_1 \geqq 0$; by a similar argument, $c_2 \leqq 0$, which contradicts the assertion $c_1 < c_2$.

(14) An optimal investment policy has no jumps other than possibly at $t = 0$.

With this result we can now investigate more closely the structure of the optimal solution. From (8) it is clear that the optimal path consists of time intervals satisfying alternately the conditions $q(t) = 0$ (shadow price and market price of capital goods are equal) and $q(t) < 0$ (shadow price of capital goods less than market price), with zero investment in the latter case. Call the intervals in which $q(t) = 0$ <u>free</u> intervals (since the non-negativity condition is not binding on those intervals), and those in which $q(t) < 0$ <u>blocked</u> intervals.

In a free interval, $q(t) = 0$ throughout the interval. Hence, $\dot{q} = 0$, or, by (10),

(15) $P_K(K_t, t) = \rho(t)$ in a free interval.

This is precisely the myopic rule discussed in section 1. In general, let us define the <u>myopic policy</u> by the equation

(16) $P_K(K_t^*, t) = \rho(t)$;

under the assumption of diminishing returns, this equation has a unique solution. Then (13) is written

(17) $K(t) = K^*(t)$ on a free interval.

But $K(t)$ is increasing. Therefore, $K^*(t)$ must be increasing

throughout any free interval. If $K^*(t)$ is a well-behaved function, it has alternately rising and falling segments. Refer to any interval which is a rising segment of the graph of $K^*(t)$ as a <u>riser</u>.

(18) A free interval lies entirely within a single riser.

Now consider any blocked interval starting at time $t_0 > 0$. It was preceded by a free interval and therefore t_0 must lie in a riser. Since neither $K(t)$ nor $q(t)$ have jumps, we must have $K(t_0) = K^*(t_0)$ and $q(t_0) = 0$. Since $I = 0$ on a blocked interval, $K(t)$ is a constant, so that $K(t) = K^*(t_0)$ for all t in the interval.

A blocked interval ending at $t_1 < +\infty$ must be followed by a free interval. By exactly parallel arguments, t_1 must lie on a riser, $K(t) = K^*(t_1)$ for all t in the blocked interval, and $q(t_1) = 0$.

If we recall that by definition $q(t) < 0$ on a blocked interval, then with the aid of (11) and (12) we can draw the following conclusions:

(19) On a blocked interval (t_0, t_1) with $t_0 > 0$, $t_1 < +\infty$,

(a) $K^*(t_0) = K^*(t_1)$;

(b) $\int_{t_0}^{t_1} \alpha(t)\, r[K^*(t_0), t]dt = 0$;

(c) $\int_{t_0}^{t} \alpha(t)\, r[K^*(t_0), t]dt < 0$ for $t_0 < t < t_1$;

(d) $\int_{t}^{t_1} \alpha(t)\, r[K^*(t_0), t]dt > 0$ for $t_0 < t < t_1$.

Relation (d) is not independent of the others but follows from (b) and (c).

Relations (b-d) have simple interpretations. Suppose it were possible to rent capital goods for some fixed period of time at a price $\rho(t)$

possibly varying in time. Then $P_K - \rho = -r$ is the instantaneous profit. Purchasing a capital good and selling it at the end of the period is exactly equivalent to renting it at a rate equal to the market rate of interest; purchasing a capital good and holding it to a point of time where the firm would wish to purchase capital goods anyway is also equivalent to renting. Then (c) assures us that it would be profitable to rent a capital good at t_0 for any term short of the full blocked interval; since in fact the firm has to buy instead of rent, and neither can it sell at time t nor does it wish to hold it then, the firm in fact does not purchase. Equation (b) says that at the margin the firm is indifferent between renting and not renting for the entire period. Relation (d) assures that the firm would not wish to rent beginning at any point in the blocked interval and ending at time t_1 (for if it did, it would buy).

There may be a blocked interval beginning at time 0 and ending at a finite time t_1; the stock of capital must be constant at $K^*(t_1)$ and $q(t_1) = 0$. However, there might be an initial jump in capital at time 0 before settling down to the blocked interval; if there is, then $q(0)$ must be zero, since investment is taking place. Then, by the same arguments,

(20) On a blocked interval $(0,t_1)$, with $t_1 < +\infty$,

(a) $K(0) \leqq K^*(t_1)$;

(b) $\int_0^{t_1} \alpha(t)\, r[K^*(t_1),t]dt \geqq 0$;

(c) strict inequality cannot hold in both (a) and (b);

(d) $\int_t^{t_1} \alpha(t)\, r[K^*(t_1),t]dt > 0, \quad 0 < t < t_1$.

To discuss blocked intervals starting at some $t_0 \geqq 0$ and continuing for all subsequent values of t, it is necessary to note the asymptotic behavior of $q(t)$. Since $p(t)$ was defined as the shadow price of capital, it is necessarily non-negative since $P_K > 0$. Hence, from the definition (6) and from (8),

$$-\alpha(t) \leqq q(t) \leqq 0.$$

However, we may certainly suppose that $\alpha(t)$ approaches zero as t approaches infinity; this would certainly be true if the interest rate were bounded away from zero or even approached zero slowly. Then,

$$(21) \qquad q(+\infty) = \lim_{t \to +\infty} q(t) = 0.$$

Consider now a blocked interval beginning at $t_0 > 0$ and continuing to positive infinity. Then $K(t)$ must be the constant $K^*(t_0)$ and $q(t_0) = 0$, so that, much like (19),

(22) On a blocked interval $(t_0, +\infty)$, with $t_0 > 0$,

(a) $\int_{t_0}^{+\infty} \alpha(t)\, r[K^*(t_0), t]dt = 0,$

(b) $\int_{t_0}^{t} \alpha(t)\, r[K^*(t_0), t]dt < 0, \quad t_0 < t.$

Finally, it is possible to have a blocked interval beginning at $t_0 = 0$ and continuing to plus infinity. It may be that the initial stock of capital, $K(0)$, is simply held intact without further investment, or it may be that there is a jump immediately to some value K, which is then never subsequently added to. In the latter case, of course, $q(0)$ must be zero since some investment has taken place. In any case, by (21), $q(+\infty) = 0$.

(23) On a blocked interval $(0, +\infty)$, $K(t)$ is a constant K, with

(a) $K(0) \leqq K$;

(b) $\int_0^{+\infty} \alpha(t)\, r(K,t)dt \geqq 0$;

(c) the strict inequality cannot hold in both (a) and (b);

(d) $\int_t^{+\infty} \alpha(t)\, r(K,t)dt > 0,\ 0 < t.$

Theorem. The optimal capital policy for a firm with irreversible investment is an alternating sequence of free and blocked intervals, constructed so as to satisfy the relevant conditions among (17-20) and (22-23).

4. Algorithmic Remarks

It may not be obvious that the stated conditions really provide a sensible way of computing the optimal policy. In particular, conditions such as (19c) and parallel conditions for the other cases refer to the values of a function at every point in the interval, and therefore the amount of trial and error needed in successive approximations to the true policy appears to be prohibitive. But in fact we are seeking only the maxima or minima of certain functions, and for these it suffices to search only among local maxima or minima; thus, (19c) asserts that the maximum value of the indicated function of time be negative and for this it suffices to calculate the integral at local maxima. The local maxima are clearly those zeros of the integrand at which its value changes from positive to negative. If the functions $P(K,t)$ and $\rho(t)$ are well-behaved, there will be only finitely many zeros in any finite period.

Recall, from (12) and (13) of section 3, that

(1) $\quad r(K,t) = \rho(t) - P_K(K,t)$

is a strictly increasing function of K for fixed t. By definition of the myopic policy,

(2) $\quad r(K_t^*,t) = 0,$

and therefore

(3) $\quad r(K,t) > 0$ if and only if $K > K^*(t)$.

Label the successive risers 1,2,... ; for well-behaved functions there are at most denumerably many risers. On any given riser, $K^*(t)$ is a strictly increasing function of t and therefore has an inverse, $t_i(K)$, the time on riser i at which $K^*(t)$ takes on the value K. The function $t_i(K)$ is defined on the range of values which $K^*(t)$ assumes on riser i; let $\underline{K}_i$ be the lower bound of this range, and $\overline{K}_i$ the upper bound.

If there is a blocked interval starting on riser i and ending on riser $j > i$, with $K(t) = K$ on that interval, then from (19a) of section 3 the blocked interval starts at $t_i(K)$ and ends at $t_j(K)$. This can, of course, only be possible if

(4) $\quad \max(\underline{K}_i, \underline{K}_j) \leqq K \leqq \min(\overline{K}_i, \overline{K}_j)$,

for if K were outside these bounds, either $t_i(K)$ or $t_j(K)$ would be undefined. In view of (19b), define, for all K satisfying (4),

(5) $\quad q_{ij}(K) = \int_{t_i(K)}^{t_j(K)} \alpha(t)\, r(K,t)dt.$

Then a second condition that must be satisfied is that

(6) $\quad q_{ij}(K) = 0.$

We now show that $q_{ij}(K)$ is strictly increasing in K within its range of definition and therefore there is at most one solution to (6). First observe that, from (2) and the definition of $t_i(K)$,

(7) $\quad r[K,t_i(K)] = 0$ for all K for which $t_i(K)$ is defined.

Then differentiate (5) with respect to K.

$$q_{ij}'(K) = \alpha[t_j(K)]r[K,t_j(K)]\, t_j'(K) - \alpha[t_i(K)]\, r[K,t_i(K)]t_i'(K)$$
$$+ \int_{t_i(K)}^{t_j(K)} \alpha(t)\, r_K(K,t)dt.$$

From (7) the first two terms vanish; since $r(K,t)$ is strictly increasing in K, $r_K > 0$ everywhere so that the last integral is positive.

(8) $\quad q_{ij}(K)$ is strictly increasing in K.

It is also useful to define

(9) $\quad q_i(K,t) = \int_{t_i(K)}^{t} \alpha(t)\, r(K,t)dt;$

(19c) of section 3 requires that $q_i(K,t) < 0$ for $t_i(K) < t < t_j(K)$. As above, we calculate

$$\partial q_i/\partial K = -\alpha[t_i(K)]\, r[K,t_i(K)]\, t_i'(K)$$
$$+ \int_{t_i(K)}^{t} \alpha(t)\, r_K(K,t)dt > 0.$$

(10) $\quad q_i(K,t)$ is strictly increasing in K for fixed t.

Also,

$$\partial q_i/\partial t = \alpha(t)\, r(K,t),$$

so that a local maximum of $q(K,t)$ as a function of time occurs at those values t_0 at which $r(K,t)$ changes sign from positive to negative as t increases. But from (3) it follows that $K^*(t) < K$ to the left of t_0, $K^*(t) > K$ to the right, so that $K^*(t)$ is increasing at t_0 and $K^*(t_0) = K$. Thus, t_0 is on a riser, say k, and $t_0 = t_k(K)$. Hence, at a local maximum, $q_i(K,t) = q_{ik}(K)$ for some k.

In the present notation the condition (19c) of section 3 is simply that $q_i(K,t) < 0$, $t_i(K) < t < t_j(K)$, for a blocked interval starting on riser i and ending on riser j. It is necessary and sufficient for this that $q_i(K,t) < 0$ at every local maximum in the same interval, and therefore that

(11) $\quad q_{ik}(K) < 0$ for all risers k, $i < k < j$, for which $q_{ik}(K)$ is defined.

We can also use (10) as additional help in screening out conceivable blocked intervals by showing that condition (11) is not satisfied. Suppose it has been shown that $q_{ik}(K) \geqq 0$. By definition this is equivalent to

$$q_i[K,\ t_k(K)] \geqq 0.$$

From (10), $q_i[K',t_k(K)] > 0$ for all $K' > K$. Consider any riser $j > k$ for which K' lies in the range of $K^*(t)$. Since the entire riser j lies beyond the entire riser k, $t_j(K') > t_k(K)$, and it has been shown that

$$q_i(K',t) > 0 \quad \text{for some } t, \quad t_i(K') < t < t_j(K'),$$

namely, for $t = t_k(K)$. There can be no blocked interval from riser i to riser j with $K(t)$ at the constant level K' for any $K' > K$.

Define now an eligible interval as one that satisfies all the necessary conditions for a blocked interval; specifically,

(12) A pair of risers i, j, with $i < j$, form an eligible interval at level K if $q_{ij}(K) = 0$ and $q_{ik}(K) < 0$ for all risers k, $i < k < j$, for which $q_{ik}(K)$ is defined.

Note that from (8) there can be at most one K for which $q_{ij}(K) = 0$, and therefore at most one eligible interval between two given risers.

The preceding remarks can be assembled to provide an algorithm for finding all possible eligible intervals:

Algorithm. Start with any given riser and consider in turn all successive risers. Suppose that we have started with riser i and reached riser j. Assume defined a number $\tilde{K}_{ij}$, to be defined recursively. Let $\overline{K}_{ij} = \min(\tilde{K}_{ij},\overline{K}_j)$, $\underline{K}_{ij} = \max(\underline{K}_i,\underline{K}_j)$. If $\overline{K}_{ij} < \underline{K}_{ij}$, then there is no eligible interval from i to j; proceed to the next riser, with $\tilde{K}_{i,j+1} = \tilde{K}_{ij}$. If $\overline{K}_{ij} \geqq \underline{K}_{ij}$, compute $q_{ij}(\overline{K}_{ij})$. If negative, again there is no eligible interval from i to j, and proceed to the next riser, with $\tilde{K}_{i,j+1} = \tilde{K}_{ij}$. If $q_{ij}(\overline{K}_{ij}) \geqq 0$, compute $q_{ij}(\underline{K}_{ij})$. If positive, again there is no eligible interval from i to j, but now we define $\tilde{K}_{i,j+1} = \underline{K}_{ij}$. Finally, if $q_{ij}(\underline{K}_{ij}) \leqq 0$, we can find K_{ij} so that $q_{ij}(K_{ij}) = 0$, with $\underline{K}_{ij} \leqq K_{ij} \leqq \overline{K}_{ij}$. Then there is an eligible interval from i to j at level K_{ij}. To continue the induction, now define $\tilde{K}_{i,j+1} = K_{ij}$. To start the procedure, define $\tilde{K}_{i,i+1} = \overline{K}_i$.

Note that $\tilde{K}_{ij}$ represents a K-value such that all higher K-values have been already excluded from consideration by the argument that $q_{ik}(\tilde{K}_{ij}) \geqq 0$

for some riser k, $i < k < j$. The computations of $q_{ij}(\overline{K}_{ij})$ and $q_{ij}(\underline{K}_{ij})$ are designed to establish the possibility that $q_{ij}(K_{ij}) = 0$ for some K_{ij}. If $q_{ij}(\overline{K}_{ij}) < 0$, then $q_{ij}(K) < 0$ for all K in the interesting range $\underline{K}_{ij}$ to $\overline{K}_{ij}$ by the monotonicity of $q_{ij}(K)$. On the other hand, if $q_{ij}(\underline{K}_{ij}) > 0$, then $q_{ij}(K) > 0$ in the relevant range; further, we now know that any value of $K > \underline{K}_{ij}$ is ruled out as the level of an eligible interval for risers beyond j.

The algorithm can easily be extended to find eligible intervals from a riser i out to infinity. We need only introduce an additional riser at infinity, with $t_\infty(K) = +\infty$ for all K, $\underline{K}_\infty = 0$ and $\overline{K}_\infty = +\infty$, and define

$$\tilde{K}_{i\infty} = \lim_{j \to +\infty} \tilde{K}_{ij} ,$$

$$q_{i\infty}(K) = \int_{t_i(K)}^{+\infty} \alpha(t)\, r(K,t)dt.$$

We must finally consider possible blocked intervals beginning at the origin. First it will be shown that there cannot be a jump at the origin to a capital stock greater than $K^*(0)$. For suppose that $K(0+0) > K^*(0)$. Since $K(t)$ is monotonic increasing, $K(t) \geqq K(0+0)$ for all $t > 0$. On the other hand, by continuity $K^*(t) < K(0+0)$ for t sufficiently close to 0, so that $K(t) > K^*(t)$ for $t > 0$ and sufficiently small, and by (3) $r(K,t) > 0$, which implies that $q(t)$ is increasing. But this is only possible if $q(0) < 0$, and therefore there was no investment at time 0.

(13) If $K(0) \geqq K^*(0)$, there is no jump at the origin; if $K(0) < K^*(0)$, the jump is to a value not exceeding $K^*(0)$.

We can now define an eligible interval from time 0 to riser j analogously to (12). First define

$$(14) \qquad q_{oj}(K) = \int_0^{t_j(K)} \alpha(t)\, r(K,t)dt.$$

Then define

(15) The origin, 0, and riser j form an <u>eligible interval</u> at level K if the following conditions are satisfied: (a) $K \geqq K(0)$; (b) $q_{oj}(K) \geqq 0$; (c) strict inequality cannot hold in both (a) and (b); $q_{ok}(K) < q_{oj}(K)$ for all k, $1 \leqq k < j$ for which $q_{ok}(K)$ is defined.

The algorithm for determining all eligible intervals beginning at the origin has two branches, according as we are considering $K = K(0)$ or $K > K(0)$. In the first case, let j_1 be the first riser, if any, for which $q_{oj}[K(0)] \geqq 0$. Having defined $j_1,\ldots,j_r$, let j_{r+1} be the first riser, if any, for which

$$q_{oj}[K(0)] > q_{oj_r}[K(0)],\ j > j_r.$$

Then each of the intervals from 0 to some j_r is eligible at level $K(0)$. In these definitions it is not excluded that one of the j_r's is the riser at infinity. Also, if the sequence of j_r's is infinite, it follows that the interval from 0 to infinity at level $K(0)$ is eligible.

For values of $K > K(0)$, the previous algorithm is fully applicable, provided we introduce a riser 0, with $t_0(K) = 0$ for all K, $\underline{K}_0 = K(0)$, and $\overline{K}_0 = K^*(0)$. Note that from (13) there can be eligible intervals from the origin at level $K > K(0)$ only if $\underline{K}_0 < \overline{K}_0$.

The optimal path, finally, is obtained by choosing eligible intervals in a mutually consistent manner. Blocked intervals are separated by free intervals, each of which must lie on a single riser. If there is a jump at the origin, we understand this to mean that there is a free interval on riser 0. With this understanding, an optimal policy is described by a finite or infinite sequence of risers, $i_1, i_2, \ldots$ which satisfy the following conditions: (a) for each r, there is an eligible interval from i_r to i_{r+1}; (b) the levels $K_{i_r i_{r+1}}$ are increasing with r; (c) if the sequence of i_r's is finite, and the last of them is a finite number, then it must be a riser which continues out to infinity (for in this case the optimal policy terminates with a free interval extending to infinity). It is understood that if the sequence of i_r's is finite, the last one may be $+\infty$, in which case it is understood that there is a terminal blocked interval.

To actually find the optimal policy after having listed the eligible intervals is a process of trial and error. One starts with an eligible interval from i_1 to i_2, say, then sees if there is an eligible interval at a higher level beginning at i_2, and continues out to infinity unless the continuation becomes impossible. If it does, we go back to some riser at which a choice of eligible intervals was possible and try a different one. There is probably no point in specifying the general algorithm more precisely; in any given concrete situation, one is apt to have considerable qualitative information about the underlying function $P(K,t)$ which can be used to guide the search more precisely.

5. The Case of Constant Returns

If we assume constant returns to capital at any given moment of time, much of the previous discussion simplifies considerably. There is, of course, the possibility of an investment policy which will yield an infinite value for the sum of discounted profits. As usual in treatments of constant returns, we exclude this case. Naturally, as in the usual finite-dimensional case, the policy of investing nothing will then be as good as any other. The only remaining question is that of listing all investment policies which will be no worse than not investing at all.

The assumption of constant returns means that $P(K,t)$ is linear in K and that P_K is a function of t alone, not of K. Then $r(K,t)$ also is independent of K and can be written $r(t)$. When $r(t) > 0$, then $K^*(t) = 0$; when $r(t) < 0$, $K^*(t) = +\infty$; when $r(t) = 0$, then $K^*(t)$ is indeterminate since all values of K are equally optimal. The time-axis is divided into intervals with $r(t) > 0$ and $r(t) < 0$, respectively, separated by points with $r(t) = 0$. Those zeros of $r(t)$ for which $r(t) > 0$ to the right and $r(t) < 0$ to the left are the analogues of the risers in the diminishing returns case; the same term will be used here. Since the analogue of a free interval becomes an interval on a vertical line, jumps at the risers are not excluded. But no investment can take place except at a riser (including as before a riser at 0).

The solution for this case can be worked out much as before except that the formulas are much simpler since the magnitudes q_{ij} are now independent of K. The final result can be put simply:

(1) The optimal policies are all those which call for jumps in capital stock at all risers i (including possibly 0) for which

$$q_{i\infty} = 0.$$

It might be useful to note that the condition that infinite profits be impossible is that $q_{i\infty} \geqq 0$ for all risers i.

6. Econometric Implications

In econometric application the function P(K,t) is not itself an observable but rather an expectation, held with subjective certainty, of future profit prospects. It is in the tradition of Professor Hicks's capital model, where actual present and planned future behavior are functions of present and anticipated prices. At any moment, under the model as given, the firm draws up an investment program for the present and future, but the only part of the program that is executed is the immediate investment decision. Hence, we observe at each moment the initial investment of a long-term investment program, with the profit function and the future course of interest rates which are believed in as of that moment. To determine the empirical implications of this model, it would be necessary to add a second relation, showing how the anticipated profit function and interest rates shift with time, possibly in response to new observations on market magnitudes.[5]

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[5] This necessarily brief statement has ignored the possibility of lags between investment decisions and investment realizations, so important in detailed empirical analysis; see Jorgenson (1965). Under conditions of subjective certainty, the calculation of the optimal policy is affected relatively little, since virtually any desired policy for investment realizations can be achieved by a suitably chosen policy for investment decisions, after some initial period. But the investment actually made at any given time will depend upon anticipations of profit functions and interest rates held at some earlier time or times. If these anticipations shift over time, due to new observations on prices and the like or for any other reason, the dependence of actual investment on observed variables will be rather complex.

This is not the place to go into the possible ways in which anticipations of profit functions and interest rates can be formed, and so we cannot develop here a complete testable version of the theoretical developments of this paper. But there is one striking and definite qualitative implication: that at any given moment either the firm is holding its desired stock of capital (as defined by the profit function of the present moment, and the current rate of interest in terms of capital goods) or there is zero gross investment.[6]

Whether or not this implication is empirically valid can be ascertained only after suitable reinterpretation of the model to apply to the available data. It need only be noted here that, loosely speaking, the firm may be expected to hold the desired stock of capital until a point of time shortly before an anticipated business cycle peak. At this point, gross investment stops abruptly. The hypothesis therefore resembles that of the flexible accelerator which works on the upswing but not on the downswing (Hicks, 1950, 44-47), but differs (a) by having a less rigid relation between the desired stock of capital and the level of output, and (b) by admitting the possibility that the collapse of investment may occur because of anticipation of the end of the boom rather than its actual occurrence.

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[6] Since any observed moment of time is the initial point of an optimal investment policy computed on the basis of the then currently held anticipations of profit functions and interest rates, there might appear to be a third possibility, that of a jump in capital to a level below that currently desired. But since the anticipations are presumably themselves shifting continuously, we do not expect desired jumps to appear.

REFERENCES

Arrow, K. J. 1962 Optimal capital adjustment. Chapter 1 in Studies in Probability and Management Science (eds. K. J. Arrow, S. Karlin, and H. Scarf), Stanford, Calif.: Stanford University Press.

Arrow, K. J. 1964 Optimal capital policy, the cost of capital, and myopic decision rules. Annals of the Institute of Statistical Mathematics, 16:21-30.

Arrow, K. J., M. Beckmann, and S. Karlin 1958 Optimal expansion of the capacity of the firm. Chapter 7 in K. J. Arrow, S. Karlin, and H. Scarf, Studies in the Mathematical Theory of Inventory and Production, Stanford, Calif.: Stanford University Press.

Arrow, K. J., T. E. Harris, and J. Marschak 1951 Optimal inventory policy. Econometrica, 19:250-72.

Bellman, R. 1957 Dynamic Programming, Princeton, N. J.: Princeton University Press.

Champernowne, D. G. 1964 Expectations and the links between the economic future and the present. In Keynes's General Theory: Reports of Three Decades (ed. R. Lekachman), London: Macmillan, 174-202.

Fisher, I. 1930 The Theory of Interest, New Haven, Conn.: Yale University Press.

Hicks, J. R. 1950 A Contribution to the Theory of the Trade Cycle, Oxford: Clarendon Press.

Jorgenson, D. W. 1965 Anticipations and investment behavior. In The Brookings Quarterly Econometric Model of the United States (eds. J. S. Duesenberry, G. Fromm, L. R. Klein, and E. Kuh), Chicago: Rand-McNally.

Keynes, J. M. 1936 The General Theory of Employment, Interest and Money, London: Macmillan.

Kurahashi, K. 1963 The optimum capacity of production facilities. Paper presented at the International Meeting of the Institute of Management Sciences, Tokyo.

Marglin, S. 1963 Approaches to Dynamic Investment Planning. Amsterdam: North-Holland.

Massé, P. 1946 Les réserves et la régulation de l'avenir, Paris: Herman, 2 vol.

Nerlove, M., and K. J. Arrow 1962 Optimal advertising policy under dynamic conditions, Economica, N.S. 29:129-42.

Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko 1962 The Mathematical Theory of Optimal Processes (English translation from the Russian), New York: Interscience Publishers.

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This paper analyzes investment policy for a firm which cannot disinvest faster than its capital depreciates. The model discussed assumes that there is a single capital good and that the firm's profit is a time-varying function of the size of the capital stock. Using methods of mathematical control theory it is shown that the optimal policy consists of alternating regimes in which positive gross investment is or is not undertaken; prices are derived which can be used as a guide to correct policies. These results are contrasted with the case where disinvestment is allowed (that is, where capital goods can be bought or sold by the firm at a fixed price). An algorithm for finding the intervals where investment will take place is presented.

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